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STABILITY ANALYSIS OF SOME DYNAMICAL MODELS FOR PRICES WITH DISTRIBUTED DELAYS

Abstract. *In the present paper, we study two nonlinear systems which describe the price dynamics of a single commodity market. The market price is expressed with the demand and supply functions for the commodity. The distributed time delay is introduced in the demand and supply prices. It is assumed that the consumer and producer behavior depends on the weighted function of the past prices. The demand can take into account the recent price and the past demand price. The producer can consider the past supply price. Thus, we obtain two nonlinear mathematical models for the price dynamics. In some conditions, there is only one positive equilibrium point. The stability of the equilibrium point and the existence of periodic solutions are investigated. We find the conditions so that the equilibrium point is locally asymptotically stable. The numerical simulations illustrate the effectiveness of our results and some conclusions and future research are provided.*

Keywords: *price dynamics, delay economic models, equilibrium point, local stability, distributed time delay.*

JEL Classification: C61, C62, D24, D5

1. Introduction

Trade cycles, business cycle and fluctuations in the price and supply of various commodities have attracted the attention of economists [20]. Often the fluctuations are caused by random factors, e.g. the weather for agricultural commodities [10]. In this direction, the cobweb model was intensively studied [2], [3], [6], [7], [10], [1], [11], [12]. Also, the Rayleigh price model with time delay has been considered in [17], [18], [19].

Mackey (1989) and Belair and Mackey (1989) give a nonlinear price adjustment model with production delay and rigorously derives a stability switching condition for which the stability of equilibrium is place and thus the stable equilibrium bifurcates to a limit cycle after the loss of stability [10]. More recently, this model was studied by Matsomoto and Nakayama [11] and Matsomoto and Szidarovszky [12], where two delays were introduced. Huang et al. [7] provided a class of economic models, where the consumer behavior is influenced not only by the instantaneous price, but also by the information regarding past prices.

Based on [7] and [1], in this paper we analyze the dynamics of the price for a single commodity market. The market price $p(t)$ is expressed with the functions $D(\cdot)$ and $S(\cdot)$, respectively, denote the demand and supply functions for the commodity. As in [1], we assume that the consumer and producer behavior depends on the weighted function of the past prices. The demand for a commodity can take into account past demand price, $p_1(t)$ and recent price, $p(t)$. The producer can consider the past supply price, $p_2(t)$. The demand and supply prices are described by the functions $k_1(s)$ and $k_2(s)$, called demand and supply price kernel, respectively. Thus, we obtain four nonlinear mathematical models for the price dynamics, where the demand and supply price weak kernels, demand and supply price Dirac kernels are highlighted.

The paper is structured as follows. In Section 2, the economic models with demand and supply price kernels are presented. Section 3 investigates the stability of the equilibrium point and the existence of the Hopf bifurcation for the model that contains the present price $p(t)$ and the past supply price $p_2(t)$. In Section 4 we analyze the model with the past demand price $p_1(t)$ and the past supply price $p_2(t)$. In Section 5 numerical simulations are given to illustrate the obtained results. Section 6 gives some conclusions and future researches.

2. Economic models with distributed delays

Considering a single commodity market, the quantity of supplied and demanded can be regarded as the function of time, namely, $G_0(t)$ and $D_0(t)$. The inventory and the level of inventory are recorded, respectively, as $S(t)$ and S_0. Let $p(t)$ be the price at time t, so that the rate of price increases in proportion to the difference between S_0 and $S(t)$ [7]:

$$\dot{p}(t) = -a(S(t) - S_0), \ a > 0, \tag{1}$$

where a is a positive real number depending on the speed of price adjustment, recording $S(t)$ as:

$$S(t) = S(0) + \int_0^t (G_0(s) - D_0(s))ds. \tag{2}$$

In the traditional cobweb model, demand function is a function of price. In [7], the demand functions is:

$$D_0(t) = a_1 - a_2 p(t) - f(p(t))\dot{p}(t), \tag{3}$$

where $a_1 > 0, \ a_2 > 0, \ a_2$ represents the sensitive degree of consumers to the increase of commodity price; $f(p(t))$ is the level of the price relying on the rate of increase and $f : R_+ \to R_+$ is a derivable function with $f'(x) > 0, \ x \in R_+$.

Generally, supply function is monotone increasing about price, but it is considered that as price goes up, the supply could not unlimitedly increase, thus the supply function is [7]:

$$G_0(t) = b_1 + b_2 g(p(t)), \tag{4}$$

where $g : R_+ \to R_+$ is given by:

$$g(x) = \frac{x}{b_3 + x} \tag{5}$$

and $b_1 > 0, \ b_2 > 0, \ b_3 > 0$.

From (1), (2) with (3) and (4), we have:

$$\ddot{p}(t) = -a(a_2 p(t) + f(p(t))\dot{p}(t) + b_2 g(p(t)) + b_1 - a_1). \tag{6}$$

In specifying how consumer behavior affects commodity demand, we assume that this behavior is governed by an integration of information regarding past price. Thus, demand for a commodity is a weighted function $p_1(t)$ of past prices. In [1], the price $p_1(t)$ is called demand price and it is defined by:

$$p_1(t) = \int_{-\infty}^t k_1(t-s)p(s)ds \tag{7}$$

where k_1 is called the demand price kernel.

The function $k_1:[0,+\infty)\to[0,+\infty)$ is piecewise continuous and it verifies ([9]):

$$\int_0^\infty k_1(s)ds=1,\ \int_0^\infty sk_1(s)ds<\infty. \tag{8}$$

In [1], the price $p_2(t)$ is called supply price and it is defined by:

$$p_2(t)=\int_{-\infty}^t k_2(t-s)p(s)ds \tag{9}$$

where k_2 is called the supply price kernel. The function $k_2:[0,+\infty)\to[0,+\infty)$ is piecewise continuous and it verifies ([9]):

$$\int_0^\infty k_2(s)ds=1,\ \int_0^\infty sk_2(s)ds<\infty. \tag{10}$$

The demand function with demand price $p_1(t)$ is given by:

$$D_1(t)=a_1-a_2p_1(t)-f(p(t))\dot{p}(t) \tag{11}$$

and the supply function with supply price $p_2(t)$ is given by:

$$G_2(t)=b_1+b_2g(p_2(t)). \tag{12}$$

Using the functions $D_0(t),\ D_1(t),\ G_0(t),\ G_2(t)$ we obtain:

the first model:

$$\dot{p}(t)=a(S_0-S_{01}(t)), \tag{13}$$

where

$$S_{01}(t)=S_{01}(0)+\int_0^t(G_0(s)-D_1(s))ds, \tag{14}$$

and the second model:

$$\dot{p}(t)=a(S_0-S_{20}(t)), \tag{15}$$

where

$$S_{20}(t) = S_{20}(0) + \int_0^t (G_2(s) - D_0(s))ds, \tag{16}$$

and the third model

$$\dot{p}(t) = a(S_0 - S_{21}(t)), \tag{17}$$

where

$$S_{21}(t) = S_{21}(0) + \int_0^t (G_2(s) - D_1(s))ds. \tag{18}$$

Using (3), (4), (11), (12), the models (13), (15), (17) are given by the following equations:

$$\ddot{p}(t) = -a(a_2 p_1(t) + f(p(t))\dot{p}(t) + b_2 g(p(t)) + b_1 - a_1). \tag{19}$$

$$\ddot{p}(t) = -a(a_2 p(t) + f(p(t))\dot{p}(t) + b_2 g(p_2(t)) + b_1 - a_1). \tag{20}$$

$$\ddot{p}(t) = -a(a_2 p_1(t) + f(p(t))\dot{p}(t) + b_2 g(p_2(t)) + b_1 - a_1). \tag{21}$$

We analyze the models (19)-(21), for the following kernels:

1. $k_1(s) = d_1 e^{-d_1 s},\ d_1 > 0,$ the demand price weak kernel;
2. $k_2(s) = d_2 e^{-d_2 s},\ d_2 > 0,$ the supply price weak kernel;
3. $k_1(s) = \delta(s - \tau_1,)\ \tau_1 > 0,$ the demand price Dirac kernel;
4. $k_2(s) = \delta(s - \tau_2,)\ \tau_2 > 0,$ the supply price Dirac kernel.

The equations (6), (19)-(21), can be rewritten as:

$$\begin{aligned}\dot{p}(t) &= q(t),\\ \dot{q}(t) &= -af(p(t))q(t) - aa_2 p(t) - ab_2 g(p(t)) + a(a_1 - b_1)\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{p}(t) &= q(t),\\ \dot{q}(t) &= -af(p(t))q(t) - aa_2 p_1(t) - ab_2 g(p(t)) + a(a_1 - b_1)\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{p}(t) &= q(t),\\ \dot{q}(t) &= -af(p(t))q(t) - aa_2 p(t) - ab_2 g(p_2(t)) + a(a_1 - b_1)\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{p}(t) &= q(t),\\ \dot{q}(t) &= -af(p(t))q(t) - aa_2 p_1(t) - ab_2 g(p_2(t)) + a(a_1 - b_1)\end{aligned} \tag{25}$$

At first, we will show that the systems (22), (23), (24), (25) have only one positive equilibrium point under some assumptions.

Proposition 1. *If* $a_1 > b_1$, *then systems (22)-(25) have only one positive equilibrium point* $(p^*, 0)$, *where* p^*, *is the positive solution of the equation:*

$$a_2 p^2 + (a_2 b_3 + b_2 + b_1 - a_1) p + b_3 (b_1 - a_1) = 0 \tag{26}$$

The proof is obtained by vanishing the right parts of the above equations.

The analysis of the local stability of the systems (22)-(25) can be done analyzing the characteristic equation of the linearized system in $(p^*, 0)$.

Systems (22) and (23) are studied in [13]. In the present paper we focus on systems (24) and (25).

3.Stability analysis and the existence of the Hopf bifurcation for (24)

With the help of the coordinate transformation

$$x(t) = p(t) - p^*, \; y(t) = q(t), \tag{27}$$

system (24) can be further rewritten as following form:

$$\begin{aligned} &\dot{x}(t) = y(t), \\ &\dot{y}(t) = -af(x(t) + p^*)y(t) - aa_2(x(t) + p^*) - ab_2 g(x_2(t) + p^*) + a(a_1 - b_1) \end{aligned} \tag{28}$$

where

$$x_2(t) = \int_{-\infty}^{t} k_2(t-s)x(s)ds \tag{29}$$

The linearized system (28) at $(0, 0)$, is:

$$\begin{aligned} &\dot{u}(t) = u_2(t), \\ &\dot{u}_2(t) = -aa_2 u_1(t) - ab_2 b_3 \beta_2 \int_{-\infty}^{0} k_2(-s)u_1(t+s)ds - a\beta_1 u_2(t) \end{aligned} \tag{30}$$

where

$$\beta_1 = f(p^*), \; \beta_2 = \frac{1}{(b_3 + p^*)^2}.$$

The characteristic equation of system (30) is given by:

$$\lambda^2 + a\beta_1\lambda + aa_2 + ab_2b_3\beta_2\int_{-\infty}^{0} k_2(-s)e^{\lambda s}ds. \qquad (31)$$

Then, we have:

Proposition 2. *If* $a_1 > b_1$, *and* k_2 *is the supply price weak kernel, the equilibrium point* $(p^*, 0)$ *is locally asymptotically stable for all* $d_2 > 0$.

Now, let k_2 be the supply price Dirac kernel. The characteristic equation of (30) is given by:

$$\lambda^2 + a\beta_1\lambda + aa_2 + ab_2b_3\beta_2 e^{-\lambda\tau_2} = 0 \qquad (32)$$

Proposition 3.(i). *If* $a_1 > b_1$, $a_2 < b_2b_3\beta_2$, *and* $\tau_2 = \tau_{21}$, *then eq. (32) has only the pure imaginary roots* $\lambda = \pm i\omega_{21}$, *where*

$$\tau_{21} = \frac{1}{\omega_{21}}\left[\arccos\left(\frac{\omega_{21}^2 - aa_2}{ab_2b_3\beta_2}\right)\right] + 2n\pi,\ n = 0,1,2,... \qquad (33)$$

and ω_{21} is a positive root of the equation:

$$\omega^4 + a^2(a\beta_1^2 - 2a_2)\omega^2 + a^2(a_2^2 - b_2^2b_3^2\beta_2^2) = 0. \qquad (34)$$

(ii) *For* τ_{21} *and* ω_{21} *given by (33) and (34), we have:*

$$\mathrm{Re}\left(\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)^{-1}\Bigg|_{\tau_2=\tau_{21},\lambda=i\omega_{21}}\right) > 0 \qquad (35)$$

Based on the above findings we have:

Theorem 4. *Assume the conditions from Proposition 3 hold. Then, we have the following results:*

1. If $\tau_2 \in [0,\ \tau_{21}]$ *all roots of (32) have negative real part. The equilibrium point* $(p^*, 0)$ *of system (23) is locally asymptotically stable.*

2. If $\tau_2 = \tau_{21}$, eq. (32) has a pair of purely imaginary roots $\pm i\omega_{21}$, all the other roots have negative real part. Thus, system (23) undergoes a Hopf bifurcation at $\tau_2 = \tau_{21}$.

4. Stability analysis and the existence of the Hopf bifurcation (25)

With the help of the coordinate transformation

$$x(t) = p(t) - p^*,\ y(t) = q(t), \tag{36}$$

system (25) can be further rewritten in the following form:

$$\begin{aligned}&\dot{x}(t) = y(t),\\ &\dot{y}(t) = -af(x(t)+p^*)y(t) - aa_2(x_1(t)+p^*) - ab_2 g(x_2(t)+p^*) + a(a_1 - b_1)\end{aligned} \tag{37}$$

with

$$x_1(t) = \int_{-\infty}^{t} k_1(t-s)x(s)ds,\ x_2(t) = \int_{-\infty}^{t} k_2(t-s)x(s)ds. \tag{38}$$

The linearized system (37) at $(0, 0)$, is:

$$\begin{aligned}&\dot{u}_1(t) = u_2(t),\\ &\dot{u}_2(t) = -a\beta_1 u_2(t) - aa_2\int_{-\infty}^{0} k_1(-s)u_1(t+s)ds - ab_2b_3\beta_2\int_{-\infty}^{0} k_2(-s)u_1(t+s)ds\end{aligned} \tag{39}$$

where

$$\beta_1 = f(p^*),\ \beta_2 = \frac{1}{(b_3 + p^*)^2}.$$

The characteristic equation of system (39) is given by:

$$\lambda^2 + a\beta_1\lambda + aa_2\int_{-\infty}^{0} k_1(-s)e^{\lambda s}ds + ab_2b_3\beta_2\int_{-\infty}^{0} k_2(-s)e^{\lambda s}ds = 0. \tag{40}$$

1.1 k_1 is demand price weak kernel, k_2 is supply price weak kernel

Proposition 5. If $a_1 > b_1$, k_1 the demand price weak kernel and k_2 the supply price weak kernel, then the characteristic equation (40) is given by:

$$\lambda^4 + q_3\lambda^3 + q_2\lambda^2 + q_1\lambda + q_0 = 0 \tag{41}$$

where

$$q_3 = a\beta_1 + d_1 + d_2,\ q_2 = a\beta_1(d_1 + d_2) + d_1 d_2,$$
$$q_1 = a(\beta_1 d_1 d_2 + a_2 d_1 + b_2 b_3 \beta_2 d_2),\ q_0 = a(a_2 + b_2 b_3 \beta_2) d_1 d_2. \tag{42}$$

The proof follows from (40) for $k_1(s) = d_1 e^{-d_1 s},\ \ k_2(s) = d_2 e^{-d_2 s}$.
From (42), we have $q_i > 0,\ i = 0,1,2,3,$ for all $d_1 > 0$ and $d_2 > 0$.
We define:

$$\Psi_{12}(d_1, d_2) = q_1 q_2 q_3 - q_1^2 - q_3^2 q_0. \tag{43}$$

Proposition 6. *Let* $a_2 > a_1\beta_1^2$ *and* $d_1 = d_{10} < \dfrac{a_2 - a\beta_1^2}{\beta_1}$. *The equation*

$$\psi_{12}(d_{10}, d_2) = 0 \tag{44}$$

has a positive root denoted by d_{20}. *For* $d_2 = d_{20}$, *the relation:*

$$\left.\frac{d\Psi_{12}(d_{10}, d_2)}{dd_2}\right|_{d_2 = d_{20}} \neq 0 \tag{45}$$

holds.

Thus, we have the following result:

Theorem 8. *If* $d_2 < d_{20}$, *then the equilibrium point* $(p^*, 0)$ *of system (25) is locally asymptotically stable. If* $d_2 = d_{20}$, *then as* d_2 *passes through* d_{20}, *a Hopf bifurcation occurs at* $(p^*, 0)$.

1.2 k_1 is demand price weak kernel, k_2 is supply price Dirac kernel

Let $a_1 > b_1$, k_1 *the demand price weak kernel and* k_2 *the supply price Dirac kernel. The characteristic equation of (39) is given by:*

$$\lambda^3 + e_2\lambda^2 + e_1\lambda + e_0 + (e_3\lambda + e_4)e^{-\lambda\tau_2} = 0 \tag{46}$$

where

$$e_2 = d_1 + a\beta_1,\ e_1 = a\beta_1 d_1,\ e_0 = a a_2 d_1, e_3 = a b_2 b_3 \beta_2,\ e_4 = a b_2 b_3 \beta_2 d_1. \tag{47}$$

Proposition 8. *If* $\tau_2 = 0$, *eq. (46) has all roots with negative real part for all* $d_1 > 0$. *If* $\tau_2 > 0$ *and* $a_2 < b_2 b_3 \beta_2$ *and* $\tau_2 = \tau_{20}$, *then eq. (46) has only a pair of pure imaginary roots* $\lambda = \pm i\omega_{20}$,
where

$$\tau_{20} = \frac{1}{\omega_{20}}\left[\arccos\left(\frac{e_4(e_2\omega_{20}^2 - e_0) + e_3\omega_{20}(\omega_{20}^3 - e_1\omega)}{e_4^2 + e_3^2\omega_{20}^2}\right)\right] + 2n\pi,\ n = 0, 1, 2, ... \quad (48)$$

and ω_{20} *is a positive root of the equation:*

$$\omega^6 + (e_2^2 - 2e_1)\omega^4 + (e_1^2 - e_3^2 - 2e_0e_2)\omega^2 + e_0^2 - e_4^2 = 0. \quad (49)$$

Moreover,

$$sign\left[\left.\frac{d\,\mathrm{Re}\,\lambda(\tau_2)}{d\tau_2}\right|_{\lambda = i\omega_{20},\ \tau_2 = \tau_{20}}\right] = sign G(\omega_{20}), \quad (50)$$

where

$$\begin{aligned} G(\omega_{20}) &= (3e_3\omega_{20}^2 - e_1e_3 + 2e_2e_4)(e_3\omega_{20}^4 + (e_2e_4 - e_1e_3)\omega_{20}^2 - e_4e_0) + \\ &+ ((2e_2e_3 - 3e_4)\omega_{20}^2 + e_1e_4)(-(e_2 - e_4)\omega_{20}^2 + e_0e_3 - e_1e_4) - e_3^2 \end{aligned}$$

Therefore, using the Hopf bifurcation Theorem ([4], [5], [8]), we have the following theorem:

Theorem 9. *Assume the conditions from Proposition 8 hold. Then, we have the following results:*

1. If $\tau_2 \in [0, \tau_{20})$ *all roots of (46) have negative real part for all* $d_i > 0, i = 1, 2$. *The equilibrium point* $(p^*, 0)$ *of system (25) is locally asymptotically stable.*

2. If $\tau_2 = \tau_{20}$ *and* $G(\omega_{20})$, *then system (25) undergoes a Hopf bifurcation at* $(p^*, 0)$.

1.3 k_1 is demand price Dirac kernel, k_2 is supply price Dirac kernel

Let $a_1 > b_1$, k_1 *the demand price Dirac kernel and* k_2 *the supply price Dirac kernel. The characteristic equation (40) is given by:*

$$\lambda^2 + a\beta_1\lambda + aa_2 e^{-\lambda\tau_1} + ab_2b_3\beta_2 e^{-\lambda\tau_2} = 0 \tag{51}$$

Proposition 10. *If* $\tau_1 = 0$ *, eq. (51) is eq. (32). For* $\tau_2^* \in [0, \tau_{21})$ *, where* τ_{21} *is given by (33),eq. (51) has all roots with negative real part. If* $\tau_1 > 0$ *and* $\tau_2 = \tau_2^*$*, then eq. (51) has only a pair of pure imaginary roots* $\lambda = \pm i\omega_{12}$ *, where*

$$\tau_{12} = \frac{1}{\omega_{12}}\left[\arccos\left(\frac{\omega_{12}^2 - ab_2b_3\beta_2\cos(\omega_{12}\tau_2^*}{aa_2}\right)\right] + 2n\pi,\ n = 0, 1, 2, \ldots \tag{52}$$

and ω_{12} *is a positive root of the equation:*

$$\begin{aligned}&\omega^4 + (2\beta_1^2a^2 - 2ab_2b_3\beta_2\cos(\omega_{12}\tau_2^*))\omega^2 - 2a^2b_2b_3\beta_1\omega\beta_2\sin(\omega\tau_2^*) + \\ &+ a^2(b_2^2b_3^2\beta_2^2 - a_2^2) = 0.\end{aligned} \tag{53}$$

Moreover,

$$sign\left[\frac{d\,\mathrm{Re}\,\lambda(\tau_1)}{d\tau_1}\right]_{\tau_2=\tau_{21},\ \lambda=i\omega_{12}} = sign G_{12}(\omega_{12})$$

where

$$\begin{aligned}G_{12}(\omega_{12}) = (&2\omega_{12}\cos(\omega_{12}\tau_{12}) + a\beta_1\sin(\omega_{12}\tau_{12}) - ab_2b_3\beta_2\tau_{21} - \\ &- ab_2b_3\beta_2\tau_2^*\sin(\omega_{12}(\tau_{12} - \tau_2^*)).\end{aligned}$$

Therefore, we have the following theorem:

Theorem 11. *Assume the conditions from Proposition 10 hold. Then, we have the following results:*

1. If $\tau_1 \in [0, \tau_{12}]$ *all roots of (61) have negative real part for* $\tau_2^* \in [0, \tau_{21}]$*. The equilibrium point* $(p^*, 0)$ *of system (25) is locally asymptotically stable.*

2. If $\tau_1 = \tau_{12}$ *and* $G_{12}(\omega_{12}) \neq 0$*, then system (25) undergoes a Hopf bifurcation at* $(p^*, 0)$.

From [13], if $a_1 > b_1$ and $\tau_2 = 0$, eq. (51) has all roots with negative real part for all $d_1 > 0$ and $\tau_1^* \in [0, \tau_{11}]$, where τ_{11} is given by:

$$\tau_{11} = \frac{1}{\omega_{11}}\left[\arccos\left(\frac{\omega_{11}^2 - a_2 b_2 b_3 \beta_2}{a a_2}\right)\right] + 2n\pi,\ n = 0, 1, 2, \ldots \tag{54}$$

and ω_{11} is a positive root of the equation:

$$\omega^4 + a(a\beta_1^2 - 2ab_2b_3\beta_2)\omega^2 + a^2(b_2^2 b_3^2 \beta_2^2 - a_2^2) = 0. \tag{55}$$

Proposition 12. *If* $\tau_2 \neq 0$ *and* $\tau_1 = \tau_1^*$, *then eq. (51) has only a pair of pure imaginary r roots* $\lambda = \pm i\omega_{21}$, *where*

$$\tau_{21} = \frac{1}{\omega_{21}}\left[\arccos\left(\frac{\omega_{21}^2 - a a_2 \cos(\omega_{21}\tau_1^*}{a b_2 b_3 \beta_2}\right)\right] + 2n\pi,\ n = 0, 1, 2, \ldots \tag{56}$$

and ω_{21} is a positive root of the equation:

$$\omega^4 + (a^2\beta_1^2 - 2aa_2\cos(\omega\tau_1^*))\omega^2 - 2a^2 a_2 b_3 \beta_1 \omega \sin(\omega\tau_2^*) + a^2(a_2^2 - b_2^2 b_3^2 \beta_2^2) = 0. \tag{57}$$

and

$$sign\left[\frac{d\,\mathrm{Re}\,\lambda(\tau_2)}{d\tau_2}\right]_{\tau_2=\tau_{21},\ \lambda = i\omega_{21}} = sign G_{21}(\omega_{21})$$

where

$$G_{21}(\omega_{21}) = (2\omega_{21}\cos(\omega_{21}\tau_{21}) + a\beta_1\sin(\omega_{21}\tau_{21}) - aa_2\tau_1^*\sin(\omega_{21}(\tau_{21} - \tau_1^*)).$$

Therefore, we have the following theorem:

Theorem 13. *Assume the conditions from Proposition 12 hold. Then, we have the following results:*

1. If $\tau_2 \in [0, \tau_{21})$ *all roots of (51) have negative real part for* $\tau_1^* \in [0, \tau_{11}]$. *The equilibrium point* $(p^*, 0)$ *of system (25) is locally asymptotically stable.*

2. If $\tau_2 = \tau_{21}$ *and* $G_{21}(\omega_{21}) \neq 0$, *then system (25) undergoes a Hopf bifurcation at* $(p^*, 0)$.

If $\tau_1 = \tau_2 = \tau$ **,** the characteristic equation (51) becomes:

$$\lambda^2 + a\beta_1\lambda + a(a_2 + b_2 b_3 \beta_2)e^{-\lambda\tau} = 0. \tag{58}$$

Proposition 14. If $\tau \neq 0$, then eq. (58) has only a pair of pure imaginary roots $\lambda = \pm i\omega_0$, where

$$\tau_0 = \frac{1}{\omega_0}\left[\arccos\left(\frac{\omega_0^2}{a(a_2 + b_2 b_3 \beta_2)}\right)\right] + 2n\pi,\ n = 0, 1, 2, \ldots \tag{59}$$

where ω_0 is a positive equation of the equation:

$$\omega^4 + a^2\beta_1^2\omega^2 - a^2(a_2 + b_2 + b_3\beta_2)^2 = 0. \tag{60}$$

Moreover,

$$sign\left[\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}\right]_{\tau_2=\tau_0,\ \lambda=i\omega_0} = sign G_0(\omega_0)$$

where

$$G_0(\omega_0) = (2\omega_0\cos(\omega_0\tau_0) + a\beta_2\sin(\omega_0\tau_0))/(\omega_0(aa_2 + ab_2b_3\beta_2).$$

The proof is obtained as in Proposition 10.

Theorem 15. *Assume the conditions from Proposition 14 hold. Then, we have the following results:*

1. If $\tau \in [0, \tau_0)$ all roots of (58) have negative real part. The equilibrium point $(p^*, 0)$ of system (25) with $\tau_1 = \tau_2 = \tau$ is locally asymptotically stable.

2. If $\tau = \tau_0$ and $G_0(\omega_0) \neq 0$, then system (25) with $\tau_1 = \tau_2 = \tau$ undergoes a Hopf bifurcation at $(p^*, 0)$.

5. Numerical Simulation

For the numerical simulations we consider the parameters: $a_1 = 125,\ a_2 = 2,$ $a = 0.08,\ b_1 = 85,\ b_2 = 180,\ b_3 = 12,$ and the functions: $f(x) = 2x + 1,\ g(x) = \frac{x}{12 + x}$.

The equilibrium state of the price is $p^* = 2.82921$.

System (24) is based on the demand function and $D_0(t) = 125 - 2p(t) - (2p(t)+1)\dot{p}(t)$, and the supply function with the supply price kernel $G_2(t) = \dfrac{85 + 180\,p_2(t)}{12 + p_2(t)}$, where $p_2 = \int_{-\infty}^{t} k_2(t-s)\,p(s)ds$.

Using formula (33) we obtain $\tau_{21} = 0.7289$. For any $\tau_2 < 0.7279$, the equilibrium point $(2.82921,\ 0)$ is locally asymptotically stable. The orbits $(t,\ p(t))$ and $(t,\ q(t))$ can be visualized in Figure 1.

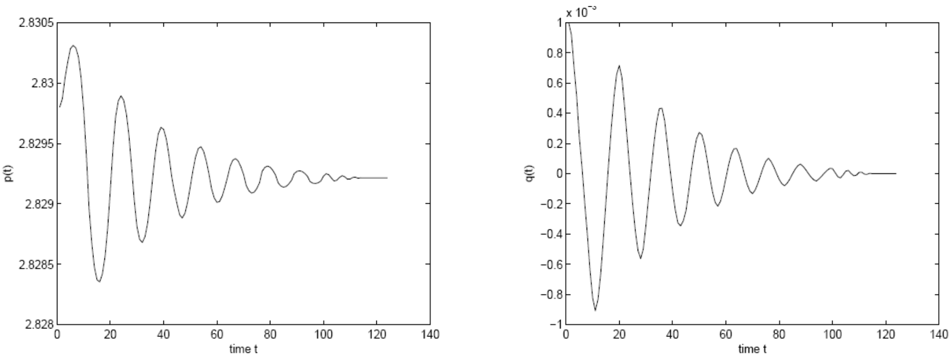

Figure 1: Trajectories of system (24) converge to the asymptotically stable equilibrium point $(2.82921,\ 0)$**, when demand function is** D_0 **and supply function is** G_2 **with supply price Dirac kernel, for** $\tau_2 = 0.5$**.**

From Theorem 4, τ_{21} is a Hopf bifurcation. In this case, for $\tau_{21} = 0.7289$ the orbit of system (24) is periodically and we have Figure 2.

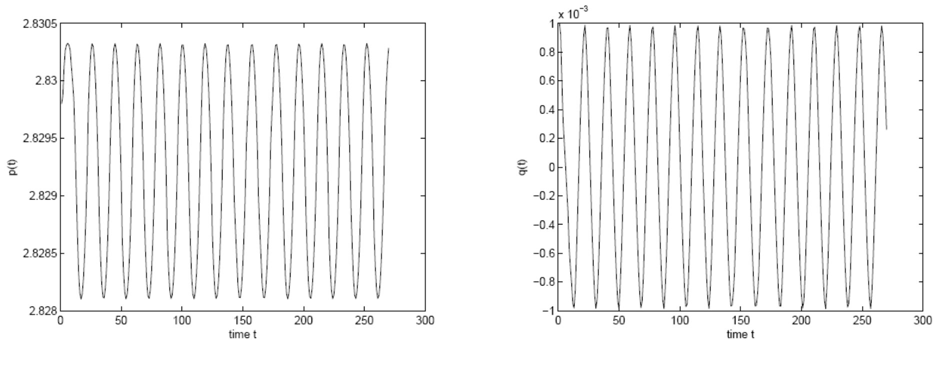

Figure 2: Trajectories of system (24) are periodically, with demand function D_0 **and supply function** G_2 **with supply price Dirac kernel, for** $\tau_2 = \tau_{21} = 0.7289$

System (25) is based on the demand function with demand price kernel $D_1(t) = 125 - 2p_1(t) - (2p(t)+1)\dot{p}(t)$, where $p_{12} = \int_{-\infty}^{t} k_1(t-s)p(s)ds$ and the supply function with the supply price kernel $G_2(t) = \frac{85 + 180p_2(t)}{12 + p_2(t)}$, where $p_2 = \int_{-\infty}^{t} k_2(t-s)p(s)ds$.

If we have the demand price weak kernel with $k_1(t) = 0.2e^{-0.2t}$, and supply price with weak kernel with $k_2(t) = d_2 e^{-d_2 t}$. From Proposition 6, we find the critical value of the parameter d_2, $d_{20} = 161.06$. From Theorem 7, the equilibrium point is locally asymptotically stable for any $d_2 < d_{20} = 161.06$. System (25) undergoes a Hopf bifurcation when $d_2 = d_{20}$. For $d_1 = 0.2$, $d_2 = 0.8$ the orbits of the price $(t,\ p(t))$ and $(t,\ q(t))$ are displayed in Figure 3.

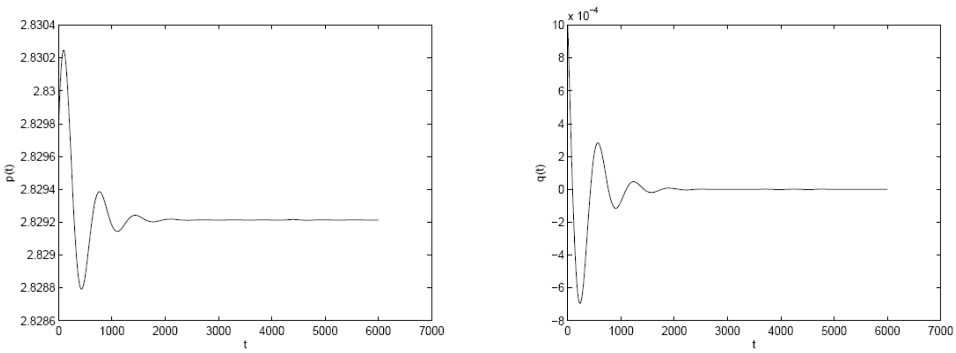

Figure 3: Trajectories of system (25) converge to the asymptotically stable equilibrium point $(2.82921,\ 0)$, where demand function D_0 has demand price weak kernel and supply function G_2 has supply price weak kernel, for $d_1 = 0.2$, $d_2 = 0.8$.

Now, we consider the demand price weak kernel with $k_1(t) = 0.8e^{-0.8t}$ and supply price with Dirac kernel $p_2(t) = p(t - \tau_2)$. From Theorem 9, the equilibrium point is locally asymptotically stable for any $\tau_2 < \tau_{20} = 0.59$. System (25) undergoes a Hopf bifurcation when $\tau_2 = \tau_{20}$. In this case, the orbits of the price $(t,\ p(t))$ and $(t,\ q(t))$ are displayed in Figure 4.

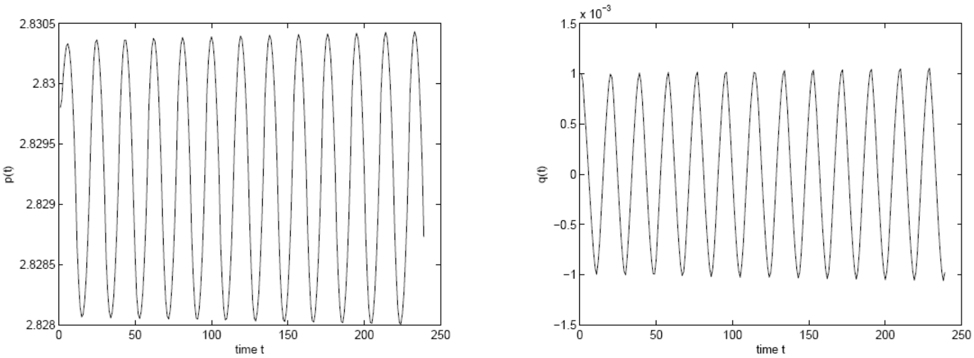

Figure 4: Trajectories of system (25) are periodically, where demand function D_1 has demand price weak kernel and supply function G_2 has supply price Dirac kernel, for $\tau_2 = \tau_{20} = 0.59$.

6. Conclusions

This paper deals with a single commodity market, where the price dynamics is studied. Taking into account the current price, the supply past price and the demand past price, four models are provided. The past prices are expressed with the Dirac or weak kernel. Therefore, two mathematical models are investigated by the stability analysis and the existence of the Hopf bifurcation.

For the first model, the current price and supply past price are considered. When we have the supply price weak kernel, the equilibrium point is locally asymptotically stable for some conditions of the parameters. In the case of the supply Dirac kernel, a Hopf bifurcation occurs, when the delay passes through a critical value.

For the second model, three subcases are analyzed: demand price weak kernel and supply price weak kernel, demand price weak kernel and supply price Dirac kernel, demand price Dirac kernel and supply price Dirac kernel. For different values of the parameters there is a cyclic behavior of the system.

For the numerical simulations we have used Maple and Matlab and the obtained figures verify the theoretical things.

A similar analysis can be carried out for the uniform distribution and the strong kernel. As in [14], [15], [16] the stochastic approach will be taken into consideration.

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